On the lowest energy excitations of one-dimensional strongly correlated electrons

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Abstract

It is proven that the lowest excitations $E_{low}(k)$ of one-dimensional half-integer spin generalized Heisenberg models and half-filled extended Hubbard models are π-periodic functions. For Hubbard models at fractional fillings $E_{low}(k+2k_f) = E_{low}(k)$, where $2k_f = \pi n$, and n is the number of electrons per unit cell. Moreover, if one of the ground states of the system is magnetic in the thermodynamic limit, then $E_{low}(k) = 0$ for any k, so the spectrum is gapless at any wave vector. The last statement is true for any integer or half-integer value of the spin.

Low-energy excitation spectra of correlated systems have long been in the center of the theoretical condensed matter research (see [1] and references therein). In 1932 Hans Bethe proposed his Ansatz method [2] for correlated wave functions, that was then used to solve many one-dimensional quantum and also two-dimensional classical models (see [3], [4], [5]). Using the Bethe Ansatz (BA) method des Cloizeaux and Pearson [6] derived the energy of low lying $S = 1$ (triplet) states of the spin-1/2 antiferromagnetic Heisenberg model (AFHM). The spectrum of the excitations was found to be $E_t(k) = 1/2J\pi \mid \sin k \mid$. $E_t(k)$ is a π-periodic function of k. des Cloizeaux and Pearson assumed the excitations $E_t(k)$ to be the elementary excitations of the system, therefore assigning to an elementary excitation spin 1. Later investigations [7] have shown that the spectrum of [6] was incomplete. The full spectrum of the $S = 1$ magnon excitations is characterized by two continuous quantum numbers. Fixing the wave vector k one still has a continuous one-parametric family of excitations. The des Cloizeaux and Pearson spectrum is only a lower bound of these excitations. As a result $S = 1$ states can not be considered as elementary indivisible particles having only one wave number k. They are rather two-particle compositions. The problem was finally clarified by Faddeev and Takhtajan in Ref. [8]. It was shown that all excitations of the spin-1/2 AFHM are superpositions of spin-1/2 elementary excitations (called kinks or spinons). For periodic rings having odd/even number of sites only odd/even number of kinks are possible in the system. Kinks do not form bound states. Their interaction manifests itself only through a scattering amplitude. Energies and momenta of many-kink states add, as for independent particles. The dispersion relation of a kink is $1/2J\pi \sin k$, with $0 \leq k \leq \pi$. The Brillouin zone is therefore only half of the original one. The $S = 1$ states of the model are pairs of kinks. The lowest excitations are found by minimizing $E(k_1) + E(k_2)$ for $k_1 + k_2 = k$. The result is $k_1 = k$ and $k_2 = 0$ (or vice versa), so the lowest excitations probe the one-spinon excitations spectrum. For the ferromagnetic case the one magnon spectrum of the one-dimensional spin-1/2 ferromagnetic Heisenberg model (FHM) has the form $2J(1 - \cos k)$, possessing a gap at nonzero k and not being π-periodic. Careful analysis shows, that one-magnon states are not the lowest energy states of the system. Magnons can form bound

states and the interaction is so strong,, that the two-magnon states have energy less than the energy of one magnon. It was shown, using Bethe Ansatz [9], that the elementary excitations of the system (excitations having one quantum number) are magnons and strings. Strings are complexes of $2M+1$ spins, having a dispersion relation $J\frac{2}{2M+1}(1-\cos k)$. The lowest bound for the spectrum of excitations is formed, when M goes to infinity. Therefore the spectrum is gapless for all wavevectors k. As will be shown below, this property characterizes *all* Heisenberg-like one-dimensional models (integer-spin or half-integer-spin), which are magnetic in the thermodynamic limit. The excitations spectrum of the half-filled one-dimensional Hubbard model was derived in [10]. The lowest-lying triplet states of the model are given by equation (16) of [10]. The function $E_t(k)$ is π-periodic and reaches its maximum at $q=\pi/2$. For a quarter filling the spectrum of the lowest energy excitations is periodic with a period of $\pi/2=2k_f$, where k_f is the Fermi wave vector of the noninteracting system. For the attractive Hubbard model the excitations spectrum is calculated in [11]. The elementary excitations of the model at half-filling are bound pairs and "free" electrons (instead of spinons and holons in the repulsive case). The energy-momentum dispersion relation of excitations is the same, as in the repulsive case, the lowest excitations being again π-periodic.

It is proven below that the π-periodicity of the lowest-lying excitations is a model independent feature and holds for all one-dimensional isotropic Heisenberg models having a half-integer value of the spin per unit cell, and for all one-dimensional isotropic Hubbard models with an odd number of electrons per unit cell. For Hubbard models at fractional fillings the spectrum of lowest excitations is periodic with a period πn where n is the number of electrons per unit cell. For systems with a nonzero groundstate magnetization we are able to prove a stronger statement. We call a 1-D system *magnetic in the thermodynamic limit*, if there exist such N_0, that, for all $N>N_0$, one has $S_N/N \geq \epsilon > 0$, where S_N is the

spin of the groundstate of a N-site ring.[1]. We prove, that if a generalized Heisenberg or Hubbard model is magnetic in the thermodynamic limit, then the spectrum of excitations is gapless at any wave vector k, i.e., that, for any k, there exists an excitation of arbitrarily small energy. The only restriction on the interaction is, that the interaction is short-ranged enough, namely, that it falls off more rapidly as $1/r$. In 1933 Felix Bloch [12] formulated his theorem stating the absence of a stationary current in the ground state of a solid with no external field. The physical justification of it is straightforward. Suppose the ground state of a solid has got a nonzero current. Let us make a closed electric circuit, consisting of the solid and some dissipative device (resistance). Then after a relaxation time the current will disappear and some amount of energy will be absorbed in the resistance. Since the energy is conserved, the final zero-current state of the solid will have energy less than the initial state. A mathematical proof of the Bloch theorem was given by Bohm in [13]. The Bloch theorem can be also formulated for the spin current, i.e., the current of the z-component of the spin. Both of the versions of the Bloch theorem have been widely used in solid state physics, for instance in the theory of superconductivity [14]. It is proven in this paper, that the spin current (current of the z-component of the spin) is zero for Heisenberg and Hubbard models in any dimensions. To prove our assertions we use the transformation introduced in [15] to prove the vanishing of the excitation gap in the thermodynamic limit of half-integer spin Heisenberg models. This transformation is an analogue of the transformation [13] for the case of lattice spin models. Consider first a one-dimensional half-integer spin Heisenberg model on a periodic ring, having $2N$ sites.

$$H = J \sum \mathbf{S}_n \cdot \mathbf{S}_{n+1} = J \sum \left[S_n^z S_{n+1}^z + \frac{1}{2} \left(S_n^+ S_{n+1}^- + S_{n+1}^+ S_n^- \right) \right] \tag{1}$$

The sign of J is not specified, therefore both ferromagnetic and antiferromagnetic cases are considered. The system is translationary invariant and the wave vector k is a good

[1]If the groundstate is nontrivially degenerate, then this condition must be satisfied for at least one of groundstates.

quantum number. For a finite ring, k takes values $\frac{2\pi}{2N}l$, where l is an integer. The lowest energy excitations of the system are defined as excitations minimizing the energy for a fixed value of k. Below we prove that the spectrum of the lowest energy excitations $E_{low}(k)$ is a π-periodic function of k, $E_{low}(k) = E_{low}(k+\pi)$. The system has full rotational symmetry, therefore it is enough to consider the excitations having $S^z_{\text{full}} = 0$. All other excitations can be produced from $S^z_{\text{full}} = 0$ by applying operators $S^+_{\text{full}}, S^-_{\text{full}}$. Consider an eigenstate $\mid \psi_k\rangle$ of H, having wave vector k and energy E.

$$T \mid \psi_k\rangle = e^{ik} \mid \psi_k\rangle, \quad H \mid \psi_k\rangle = E \mid \psi_k\rangle \tag{2}$$

Here T is the translation operator, $n \to n+1$. Let us write the state $\mid \psi_k\rangle$ as a linear combination of $S^z_{\text{full}} = 0$ spin configurations $\psi_k = \sum_\sigma A_\sigma \mid \sigma\rangle$.

Any $S^z_{\text{full}} = 0$ configuration is characterized by $2NS$ numbers $x_1 \le x_2 \le x_3 ...$ showing the positions of fictitious particles, each of them increasing S^z on a given site by one. The vacuum of the system is assumed to have all spins down. Since the system is periodic, the x's are defined only *mod* $2N$. Consider the state

$$\psi_{k+\pi} = \sum_\sigma e^{\frac{2\pi i}{2N}(x_1+x_2+...)} A_\sigma \mid \sigma\rangle. \tag{3}$$

Adding $2N$ to any of the x's does not change the $\psi_{k+\pi}$. Therefore the mentioned freedom of defining the x's is satisfied. If the operator T acts on the state $\psi_{k+\pi}$, then all x's are incremented by 1. Therefore an additional phase factor $e^{i\pi 2S} = e^{i\pi}$ is acquired (2S is odd), and the state $\psi_{k+\pi}$ has a wave vector $k+\pi$. In the same fashion the state

$$\psi_{k-\pi} = \sum_\sigma e^{-\frac{2\pi i}{2N}(x_1+x_2+...)} A_\sigma \mid \sigma\rangle \tag{4}$$

has a wave vector $k-\pi$. Note that $k+\pi$ and $k-\pi$ are the same wave vectors since they differ by 2π. Let us evaluate the expectation value $\langle \psi_{k+\pi} \mid H \mid \psi_{k+\pi}\rangle$. The Ising part of the Heisenberg Hamiltonian does not flip spins. In contrast the exchange part may change the position of one spin-up fictitious particle, which amounts to a phase factor $e^{\pm\frac{2\pi i}{2N}}$, depending on a direction of the move. This explains the result of a formal calculation, which gives:

$$\langle\psi_{k+\pi} \mid H \mid \psi_{k+\pi}\rangle = \quad (5)$$

$$= \sum_n J\left[1/2\left(e^{\frac{2\pi i}{2N}}\langle\psi_k \mid S_n^+ S_{n+1}^- \mid \psi_k\rangle + e^{-\frac{2\pi i}{2N}}\langle\psi_k \mid S_{n+1}^+ S_n^- \mid \psi_k\rangle\right) + \langle\psi_k \mid S_{n+1}^z S_n^z \mid \psi_k\rangle\right].$$

To understand the physical meaning of the operator $S_n^+ S_{n+1}^- - S_{n+1}^+ S_n^-$ let us find the time derivative of the S_n^z. Due to basic principles of quantum mechanics one has [16]

$$\frac{d}{dt}S_n^z = \frac{i}{\hbar}[H, S_n^z] = \frac{iJ}{2\hbar}\left(S_{n-1}^+ S_n^- - S_n^+ S_{n-1}^-\right) - \frac{iJ}{2\hbar}\left(S_n^+ S_{n+1}^- - S_{n+1}^+ S_n^-\right) = \mathcal{J}_n^z - \mathcal{J}_{n+1}^z \quad (6)$$

where $\mathcal{J}_n^z = \frac{iJ}{2\hbar}\left(S_{n-1}^+ S_n^- - S_n^+ S_{n-1}^-\right)$. Therefore the operator $\mathcal{J}_n^z = \frac{iJ}{2\hbar}\left(S_{n-1}^+ S_n^- - S_n^+ S_{n-1}^-\right)$ has a meaning of the current of the z component of the spin through the bond $n-1, n$. The time derivative of S_n^z is then a difference of incoming and outcoming currents. For any eigenstate of the Hamiltonian H all expectation values are stationary, therefore $\frac{d}{dt}\langle S_n^z\rangle = \langle\mathcal{J}_n^z\rangle - \langle\mathcal{J}_{n+1}^z\rangle = 0$. As a result for any eigenstate of H the expectation value $\langle\mathcal{J}_n^z\rangle$ does not depend on n. The spin-current operator $\vec{\mathcal{J}}_n$ is a pseudovector with components $\mathcal{J}_n^x, \mathcal{J}_n^y, \mathcal{J}_n^z$.

For the states having a definite value of S_{full}^z it is

$$\langle\mathcal{J}_n^x\rangle = \langle\mathcal{J}_n^y\rangle = 0. \quad (7)$$

This is the selection rule for vector operators discussed in §29 of [16]. Taking a thermodynamic limit of (5) one finds

$$\langle\psi_{k+\pi} \mid H \mid \psi_{k+\pi}\rangle - \langle\psi_k \mid H \mid \psi_k\rangle = 2N\frac{J}{2}(e^{-\frac{2\pi i}{2N}}\langle\psi_k \mid S_{n+1}^+ S_n^- \mid \psi_k\rangle + e^{\frac{2\pi i}{2N}}\langle\psi_k \mid S_n^+ S_{n+1}^- \mid \psi_k\rangle - \quad (8)$$

$$-\langle\psi_k \mid S_{n+1}^+ S_n^- \mid \psi_k\rangle - \langle\psi_k \mid S_n^+ S_{n+1}^- \mid \psi_k\rangle) = iJ\pi\left(\langle\psi_k \mid S_n^+ S_{n+1}^- \mid \psi_k\rangle - \langle\psi_k \mid S_{n+1}^+ S_n^- \mid \psi_k\rangle\right).$$

In exactly the same way:

$$\langle\psi_{k-\pi} \mid H \mid \psi_{k-\pi}\rangle - \langle\psi_k \mid H \mid \psi_k\rangle = -iJ\pi\left(\langle\psi_k \mid S_n^+ S_{n+1}^- \mid \psi_k\rangle - \langle\psi_k \mid S_{n+1}^+ S_n^- \mid \psi_k\rangle\right). \quad (9)$$

Note that the expressions (8), (9) differ by sign. Therefore one of the states (3), (4) has an expectation value of the energy lower or equal than the state $\mid \psi_k\rangle$. In other words, having some excitation at the wave vector k one can always construct an excitation at the wave vector $k+\pi$ having less or equal energy. Let us now take ψ_k to be a minimum

energy excitation for a given value of the wave vector k. As has been proven there exists an excitation with a wave vector $k+\pi$ having the energy less or equal to the energy of ψ_k. Therefore: $E_{low}(k) \geq E_{low}(k+\pi)$. But we could have also started from the state $k+\pi$ to show that $E_{low}(k+\pi) \geq E_{low}(k)$. Therefore $E_{low}(k) = E_{low}(k+\pi)$, so the spectrum of the lowest energy excitations is π-periodic. Equal sign in the inequalities above are realized if and only if $\langle \mathcal{J}_n^z \rangle = 0$ for any state $\psi_{low}(k)$. Combining this fact with (7) one has $\vec{\mathcal{J}}_n = 0$, for the lowest excitations of H having $S^z_{\text{full}} = 0$. Now consider integer spin models. Then the transformations (3), (4) produce states with the same wave vector as the initial state. Therefore the π-periodicity does not work. The statement, that the spin current is equal to zero for the lowest-lying $S^z_{\text{full}} = 0$ states is still valid (otherwise we would find a state with the same k having lower energy). For the spin-1/2 AFHM case it is easy to understand, why the spin current is equal to zero for the lowest-lying states in the thermodynamic limit. As has been already said, the lowest-lying excitations are two-spinon excitations, so they involve *two* spin-1/2 particles. To produce a nonzero spin current through each bond in the thermodynamic limit one would need to excite a finite number of the particles *per unit volume(length).*

Inclusion of more than nearest-neighbor interactions or next order terms like $(\mathbf{S}_n \cdot \mathbf{S}_{n+1})^2$ does not change anything in the proof. Let us, for instance, add to the Hamiltonian (1) a next-nearest-neighbor interaction term $\alpha(\mathbf{S}_n \cdot \mathbf{S}_{n+2})$. Here α is a coupling constant. Then the equations (2)-(4) are not changed. The right-hand side of the equation (5) acquires an additional term

$$\alpha \sum_n \left[1/2 \left(e^{2\frac{2\pi i}{2N}} \langle \psi_k \mid S_n^+ S_{n+2}^- \mid \psi_k \rangle + e^{-2\frac{2\pi i}{2N}} \langle \psi_k \mid S_{n+2}^+ S_n^- \mid \psi_k \rangle \right) + \langle \psi_k \mid S_{n+2}^z S_n^z \mid \psi_k \rangle \right]. \quad (10)$$

This term is obtained by a direct calculation, and its physical explanation is similar to the explanation of the equation (5). The Ising part of the next-nearest-neighbor term $\alpha(\mathbf{S}_n \cdot \mathbf{S}_{n+2})$ does not flip spins. In contrast to that, the exchange part may change the position of one spin-up fictitious particle by *two* lattice sites, which amounts to a phase $e^{\pm 2\frac{2\pi i}{2N}}$, depending on a direction of the move. Taking the thermodynamic limit, one has an

additional term

$$2i\alpha\pi\left(\langle\psi_k \mid S_n^+ S_{n+2}^- \mid \psi_k\rangle - \langle\psi_k \mid S_{n+2}^+ S_n^- \mid \psi_k\rangle\right). \tag{11}$$

on the right-hand side of the equation (8). In the same fashion the equation (9) acquires the term

$$-2i\alpha\pi\left(\langle\psi_k \mid S_n^+ S_{n+2}^- \mid \psi_k\rangle - \langle\psi_k \mid S_{n+2}^+ S_n^- \mid \psi_k\rangle\right). \tag{12}$$

Then arguments, identical to those given after equation (9), lead to a π-periodicity of lowest-lying excitations. One can now proceed adding next-next-nearest-neighbor interactions, etc. The proof goes exactly in the same way, as soon as the interaction range is *finite*. If the interaction has a long range behavior, then one must require the long-range part to fall off quickly enough. Namely, assuming that the interaction falls off as $1/r^{1+\gamma}$, $\gamma > 0$, one can successfully repeat the steps (2)-(9) to prove the π-periodicity conjecture in this case. I leave a detailed proof for this case to a longer paper [17]. One can also add nonlinear terms like $(\mathbf{S}_n \cdot \mathbf{S}_{n+1})^m$, and, repeating the same steps, show by a direct calculation, that the π-periodicity conjecture holds for this case too. Let us, for instance, add to the Hamiltonian (1) a term $\beta\,(\mathbf{S}_n \cdot \mathbf{S}_{n+1})^2$. We can rewrite this term as $\left[S_n^z S_{n+1}^z + \frac{1}{2}\left(S_n^+ S_{n+1}^- + S_{n+1}^+ S_n^-\right)\right]^2$. This expression contains terms which do not change x_i's, terms that change one of x_i's by 1, and terms, like $(S_{n+1}^+ S_n^-)^2$, that change positions of *two* x_i's by one. An interested reader might check again, that the steps (2)-(9) can be straightforwardly repeated.

To formulate it shortly, for any one-dimensional half-integer-spin per unit cell isotropic Heisenberg Hamiltonian the π-periodicity conjecture is true. The proof of the general conjecture goes exactly in the same way, as the examples considered above, and will be given in the longer paper [17]. One can also note, that the full spherical symmetry is not necessary. The only facts really used are, that the Hamiltonian is symmetric with respect to a rotation around the z-axis, and that the lowest energy excitations can be chosen to have $S^z_{\text{full}} = 0$. As long as these conditions are satisfied, the spectrum of the lowest energy excitations is π-periodic.

As have already been stated, the proof works also for long-range interactions, if the interaction goes to zero more rapidly than $1/r$. Haldane and Shastry introduced an integrable Heisenberg model with an interaction going as $1/r^2$. The spectrum of lowest spinon excitations of this model is given in [18], and is π-periodic, which is an example of the π-periodicity conjecture, considered here.

Let us note, that in some models not only the lowest excitations might be π-periodic, but the groundstate itself can break the discrete translational symmetry and dimerize. In this case the size of the unit cell is doubled, and the Brillouin zone is halved. Then *all* the excitations of the system will possess a π-periodicity, which is in this case related to a symmetry breaking and dimerization. An example of such a system is a frustrated 1-D spin-one-half Heisenberg chain with $J_2 \gg J_1$, where J_1, J_2 are antiferromagnetic nearest-neighbor and next-nearest-neighbor couplings. Extensive studies of this model were performed in [19]. This model may also be viewed as a zigzag chain, with an intrachain coupling J_2 and an interchain coupling J_1. The model is exactly solvable for $J_2 = 1/2J_1$ [20], the groundstate is dimerized, and the discrete translational symmetry is broken. Density Matrix Renormalization Group studies ([19]) show, that the discrete symmetry breaking appears not only in the exactly solvable case, but for all $J_2 > 0.25J_1$. There are also helical correlations in the model, which could lead to an incommensurate order in higher-dimensions, but which are destroyed in one dimension, leading to a finite correlation length and to a gapped state.

Let us now turn to Hubbard models. In this case we claim the following $2k_f$-periodicity conjecture to be true. For any isotropic one-dimensional extended Hubbard model the spectrum of lowest-lying excitations is $2k_f$-periodic, where $2k_f = \pi n \mod 2\pi$, and n is the number of electrons per unit cell. Let us note, that for many simple one-dimensional systems this definition of k_f coincides with the Fermi wave vector, when the hubbard U is switched off, but for more complicated systems (for instance for systems with partially filled bands), it is not the case. Consider first the simplest case of a one-band Hubbard model

$$H = t \sum_{i,\sigma} (c^{\dagger}_{i,\sigma} c_{i+1,\sigma} + h.c.) + U \sum_{i} (n_{i,\uparrow} n_{i,\downarrow}) \quad . \tag{13}$$

having n electrons per site, n is a rational number, $n = c_1/c_2$, where c_1 and c_2 are integers. We consider a ring of $2c_2N$ sites, eventually taking a limit $N \to \infty$. The number of electrons on the ring is equal to $2c_1N$ and is an even number, therefore the full spin of the system takes integer values. Because of the full rotational symmetry, it is enough to consider only the states $S^z_{\text{full}} = 0$. All other excitations can be produced from $S^z_{\text{full}} = 0$ by applying operators $S^+_{\text{full}}, S^-_{\text{full}}$. Analogously to (2) we consider an eigenstate $\mid \psi_k\rangle$ of H, having a wave vector k and energy E. We write the state $\mid \psi_k\rangle$ as a linear combination of $S^z_{\text{full}} = 0$ electron configurations $\psi_k = \sum_\sigma A_\sigma \mid \sigma\rangle$. Each of the configurations $\mid \sigma\rangle$ is characterized by c_1N numbers $x_1 < x_2 < x_3 ...$, showing the positions of c_1N spin-up electrons, and by c_1N numbers $x'_1 < x'_2 < x'_3 ...$, showing the positions of spin-down electrons. Since the system is periodic, these positions are defined only *mod* $2Nc_2$. Now, in analogy to (3), consider the state $\psi_{k+2k_f} = \sum_\sigma e^{\frac{2\pi i}{2Nc_2}(x_1+x_2+...)} A_\sigma \mid \sigma\rangle$. Adding $2Nc_2$ to any of the x's does not change the $\psi_{k+\pi}$. Therefore the mentioned freedom of defining the x's is satisfied. If the operator T acts on the state $\psi_{k+\pi}$, then all x's are incremented by 1. Therefore an additional phase factor $e^{i\pi c_1/c_2} = e^{i2k_f}$ is acquired, and the state ψ_{k+2k_f} has a wave vector $k + 2k_f$. Now we need to construct a state, analogous to the state (4). Since $k + 2k_f$ and $k - 2k_f$ are in general *different* wave vectors, simply taking the state $k - 2k_f$ will not work. But we note, that since n is rational, there always exist an integer number X, such that $k - X2k_f$ and $k + 2k_f$ are the same wave vectors. Let us take $X = 2c_2 - 1$. Then $k - X2k_f = k - (2c_2 - 1)\pi c_1/c_2 = k - 2\pi c_1 + \pi c_1/c_2 = k + \pi c_1/c_2 = k + 2k_f$. Consider now the state $\psi_{k-X2k_f} = \sum_\sigma e^{-X\frac{2\pi i}{2Nc_2}(x_1+x_2+...)} A_\sigma \mid \sigma\rangle$. This state has a wave vector $k - X2k_f$, which is, as we already said, the same as $k + 2k_f$.

Let us now evaluate the expectation value $\langle \psi_{k+2k_f} \mid H \mid \psi_{k+2k_f}\rangle$. The interaction U, and the spin-down hopping do not change the positions of spin-up electrons. In contrast the spin-up hopping may change the position of one spin-up electron by one, which amounts to a phase factor $e^{\pm\frac{2\pi i}{2c_2N}}$, depending on a direction of the move. This explains the result of a direct calculation, which gives, that in the thermodynamic limit $N \to \infty$ one has

$$\langle\psi_{k+2k_f} \mid H \mid \psi_{k+2k_f}\rangle - \langle\psi_k \mid H \mid \psi_k\rangle == it\pi \left(\langle\psi_k \mid c^+_{n,\uparrow}c_{n+1,\uparrow} \mid \psi_k\rangle - \langle\psi_k \mid c^+_{n+1,\uparrow}c_{n,\uparrow} \mid \psi_k\rangle\right). \quad (14)$$

$$\langle\psi_{k-X2k_f} \mid H \mid \psi_{k-X2k_f}\rangle - \langle\psi_k \mid H \mid \psi_k\rangle == -Xit\pi \left(\langle\psi_k \mid c^+_{n,\uparrow}c_{n+1,\uparrow} \mid \psi_k\rangle - \langle\psi_k \mid c^+_{n+1,\uparrow}c_{n,\uparrow} \mid \psi_k\rangle\right). \quad (15)$$

We see, that, since X is by construction a positive integer, if one of the above expressions is positive, the other is negative, and vice versa. Therefore one of the above states has the expectation value of energy less or equal to the energy of the state $\mid \psi_k\rangle$. Note that both of considered states have the wave vector $k + 2k_f$. Now repeating arguments given after the expression (9), we state that the lowest excitations are periodic functions with a period $2k_f$. The $2k_f$-conjecture is therefore proven. It is again possible to check, that adding more than nearest-neighbor hoppings does not change the picture. One can also add arbitrary intersite interaction terms, of the type $V_k n_i n_{i+k}$, since these terms are diagonal in the considered basis and drop out of calculations. One can also consider 1-D lattices with more than one site per unit cell (for instance n-leg ladders), to show again, that the spectrum of excitations is $2k_f$ periodic, where $2k_f$ we define as πn, n being the number of electrons per unit cell.

So far we used for our derivations the subspace $S^z_{\rm full} = 0$. Consider now a Heisenberg model, where one of the (possibly many) ground states is magnetic in the thermodynamic limit,i.e., $\lim_{N\to\infty} S_{\rm full}/2N \geq M > 0$, where $S_{\rm full}$ is the spin of the ground state. Then for the fixed N the $S^z_{\rm full}$ of the ground state can be chosen to be from $-2MN$ to $2MN$. Let us choose the ground state to have $S^z_{\rm full} = C$, and write it in a linear combination of $S^z_{\rm full} = C$ configurations. Let us now consider the state $\mid \psi'\rangle = \sum_\sigma e^{G\frac{2\pi i}{2N}(x_1+x_2+...)}A_\sigma \mid \sigma\rangle$, where $x_1, x_2, ...$ are again the positions of the $2NS + 2C$ fictitious particles. G is an integer number. The wave vector of the state $|\psi' >$ differs from the wave vector of the ground state by $2\pi G(S + C/N)$. Now choosing different (allowed) values of C and G, one can show after a careful analysis (which will be given in details in [17]), that in the thermodynamic limit there is no gap at any value of k. This result does not depend on whether the spin is half-integer or integer. Therefore if the system is magnetic in the thermodynamic limit, the spectrum is gapless for any k. A similar result can be obtained for one-dimensional extended Hubbard models [17].

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